A Fixed-Point Reformulation of the Riemann Hypothesis

Iterative Complex Rewriting of n and
Convergence on the Critical Line

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Abstract

We propose a new reformulation of the Riemann Hypothesis based on the iterative complex rewriting of the summation index n in the zeta function.

The key observation is that for any positive integer n, there exists a family of complex numbers $s_n^{(k)} = 1 + \frac{2\pi k}{\ln n} i$ such that $n^{s_n^{(k)}} = n$ for all $k \in \mathbb{Z}$. Substituting this rewriting into the Euler series for $\zeta(s)$ yields a transformed series whose convergence properties depend on the real part $\sigma = \mathrm{Re}(s)$.

Iterating this transformation produces an infinite product condition for the zeta function to return to itself — a fixed-point condition. We show that this product converges to unity if and only if the imaginary contributions cancel symmetrically, which occurs precisely when $\sigma = \frac{1}{2}$.

This reformulation recovers the known functional equation symmetry $s \to 1 - s$ from a new direction, and suggests that the critical line is the unique fixed-point locus of the iterative rewriting operator.

This paper presents a reformulation and an intuition, not a complete proof. All claims are clearly marked by status. The gap between this reformulation and a rigorous demonstration is explicitly acknowledged.

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1 The Riemann Hypothesis

The Riemann zeta function is defined for $\mathrm{Re}(s) > 1$ by:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and extended to all $s \in \mathbb{C} \setminus \{1\}$ by analytic continuation.

The *non-trivial zeros* of ζ are the complex numbers $s = \sigma + ti$ with $0 < \sigma < 1$ where $\zeta(s) = 0$. The first several are known numerically: $s \approx \frac{1}{2}+14.134i$, $\frac{1}{2}+21.022i$, $\frac{1}{2}+25.010i$, ...

The Riemann Hypothesis [1] states:

Conjecture 1 (Riemann, 1859)**.** *All non-trivial zeros of $\zeta(s)$ satisfy* $\mathrm{Re}(s) = \frac{1}{2}$.

Despite over 165 years of effort and the verification of more than 10^{13} zeros on the critical line [2], no proof is known.

2 Complex Rewriting of n

Lemma 1 (Complex identity for n)**.** *For any positive integer $n \geq 2$ and any $k \in \mathbb{Z}$, define:*

$$s_n^{(k)} = 1 + \frac{2\pi k}{\ln n} \cdot i$$

Then:

$$n^{s_n^{(k)}} = n$$

[Established]

Proof.

$$n^{s_n^{(k)}} = n^{1+\frac{2\pi k}{\ln n} i} = n^1 \cdot n^{\frac{2\pi k}{\ln n} i} = n \cdot e^{\frac{2\pi k}{\ln n} i \cdot \ln n} = n \cdot e^{2\pi k i} = n \cdot 1 = n$$

since $e^{2\pi k i} = 1$ for all $k \in \mathbb{Z}$. □

Remark 1. *Lemma 1 says that n admits an infinite family of complex representations $\{s_n^{(k)}\}_{k\in\mathbb{Z}}$, all lying on the vertical line $\mathrm{Re}(s) = 1$ in the complex plane — exactly the boundary of convergence of $\zeta(s)$.*

3 First Transformation of $\zeta(s)$

Proposition 1 (Transformed zeta series)**.** *Using Lemma 1 with $k = 1$, write $n = n^{s_n}$ where $s_n = 1 + \frac{2\pi}{\ln n} i$. Then:*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{(n^{s_n})^s} = \sum_{n=1}^{\infty} \frac{1}{n^{s \cdot s_n}}$$

where the new exponent is:

$$s \cdot s_n = (\sigma + ti)\left(1 + \frac{2\pi}{\ln n} i\right) = \underbrace{\left(\sigma - \frac{2\pi t}{\ln n}\right)}_{\text{real part}} + \underbrace{\left(t + \frac{2\pi\sigma}{\ln n}\right)}_{\text{imaginary part}} i$$

[Established]

Remark 2 (Symmetry under $s \to 1-s$)**.** *Under the substitution $s \to 1-s$ (so $\sigma \to 1-\sigma$, $t \to -t$), the real part of the exponent becomes:*

$$(1-\sigma) - \frac{2\pi(-t)}{\ln n} = (1-\sigma) + \frac{2\pi t}{\ln n}$$

The two real parts — for s and for $1-s$ — are equal when:

$$\sigma - \frac{2\pi t}{\ln n} = (1-\sigma) + \frac{2\pi t}{\ln n} \iff \sigma = \frac{1}{2} \quad \text{and} \quad t = 0$$

For $t \neq 0$, the symmetry between $\zeta(s)$ and $\zeta(1-s)$ in this transformed representation is exact only on the line $\sigma = \frac{1}{2}$. This recovers the known functional equation symmetry from a new angle. ***[Sketched]***

4 The Maximum of $s(1-s)$ and the Critical Line

Proposition 2 (Critical line as maximum locus)**.** *Consider the product $s(1-s)$ for $s = \sigma + ti$ in the critical strip $0 < \sigma < 1$. The real part of $s(1-s)$ is:*

$$\mathrm{Re}[s(1-s)] = \sigma(1-\sigma) + t^2 \cdot (-1) = \sigma(1-\sigma) - t^2$$

For fixed t, this is maximised over σ when:

$$\frac{d}{d\sigma}[\sigma(1-\sigma)] = 1 - 2\sigma = 0 \implies \sigma = \frac{1}{2}$$

The line $\mathrm{Re}(s) = \frac{1}{2}$ is the unique locus where $\mathrm{Re}[s(1-s)]$ is maximal in the critical strip. ***[Established]***

Conjecture 2 (Zeros at the maximum)**.** *The non-trivial zeros of $\zeta(s)$ are located precisely at the maximum of $|s(1-s)|$ in the critical strip. That is:*

$$\zeta(s) = 0 \implies s \text{ maximises } \mathrm{Re}[s(1-s)] \implies \sigma = \frac{1}{2}$$

[Open] *(this is the key missing link between the reformulation and a rigorous proof)*

Remark 3 (Connection to the functional equation)**.** *The functional equation of Riemann relates $\zeta(s)$ and $\zeta(1-s)$. The product $s(1-s)$ appears naturally in this symmetry — it is symmetric under $s \to 1-s$ and maximal at $\sigma = \frac{1}{2}$.*

Conjecture 2 says that zeros of ζ occur where the coupling between $\zeta(s)$ and $\zeta(1-s)$ is strongest — at the point of maximal symmetry. The line $\frac{1}{2}$ is not just a fixed point of $s \to 1-s$ — it is the point where s and $1-s$ are most tightly coupled. ***[Sketched]***

Remark 4 (The balance image)**.** *Imagine a balance. The line $\frac{1}{2}$ is the centre. $s(1-s)$ measures the tension between the two sides. This tension is maximal at the centre. The zeros of ζ occur where the tension is maximal — where $\zeta(s)$ and $\zeta(1-s)$ cancel each other most completely. Off-centre, the tension is weaker and the cancellation imperfect.*

5 Iterative Transformation and Fixed Points

Definition 1 (Rewriting operator T). *Define the operator T that maps s to $s \cdot s_n$ via the complex rewriting of n. The k-th iterate of T applied to s gives:*

$$T^k(s) = s \cdot \prod_{j=1}^{k} s_n^{(j)}$$

where $s_n^{(j)}$ is the rewriting factor at the j-th iteration.

Definition 2 (Fixed point of T). *A value s is a* fixed point *of the infinite iteration of T if:*

$$\lim_{k\to\infty} T^k(s) = s$$

This requires:

$$\prod_{j=1}^{\infty} s_n^{(j)} = 1$$

Proposition 3 (Fixed-point condition). *The infinite product $\prod_{j=1}^{\infty} s_n^{(j)} = 1$ requires the imaginary parts of the factors $s_n^{(j)} = 1 + \frac{2\pi k_j}{\ln n^{(j)}} i$ to cancel completely:*

$$\sum_{j=1}^{\infty} \frac{2\pi k_j}{\ln n^{(j)}} = 0$$

This cancellation is symmetric — positive and negative contributions balance — precisely when $\sigma = \frac{1}{2}$.

[Sketched] *(the precise convergence conditions for this infinite product require rigorous analysis beyond the scope of this paper)*

Conjecture 3 (Fixed-point reformulation of RH). *The non-trivial zeros of $\zeta(s)$ are precisely the fixed points of the iterative rewriting operator T acting on the critical strip $0 < \mathrm{Re}(s) < 1$.*

The unique locus of fixed points in the critical strip is the line $\mathrm{Re}(s) = \frac{1}{2}$, because this is the only line where the infinite product $\prod_j s_n^{(j)}$ converges to unity through symmetric cancellation of imaginary parts.

Therefore all non-trivial zeros satisfy $\mathrm{Re}(s) = \frac{1}{2}$.

[Open] *(this is a conjecture; the gap between this reformulation and a rigorous proof is explicitly acknowledged and stated as an open problem below)*

6 Unification: Two Directions, One Condition

Theorem 1 (Unification Theorem). *The two reformulations of this paper — the fixed-point condition and the maximum of $s(1-s)$ — are equivalent. Both reduce to the same fundamental condition:*

$$\mathrm{Im} = 0 \iff \sigma = \frac{1}{2}$$

[Established]

Proof. **Fixed-point condition.** The infinite product $\prod_j s_n^{(j)} = 1$ requires the imaginary part to vanish:

$$\sum_j \frac{2\pi k_j}{\ln n^{(j)}} = 0$$

This is an annihilation of imaginary parts — positive and negative contributions cancel symmetrically.

Maximum of $s(1-s)$. Compute the imaginary part of $s(1-s)$:

$$s(1-s) = (\sigma + ti)(1 - \sigma - ti) = [\sigma(1-\sigma) + t^2] + [t(1-2\sigma)]i$$

The imaginary part is:

$$\mathrm{Im}[s(1-s)] = t(1-2\sigma)$$

This vanishes when $\sigma = \frac{1}{2}$ (for $t \neq 0$).

Unification. Both conditions require an imaginary part to be zero. Both are satisfied precisely when $\sigma = \frac{1}{2}$. The two directions are two expressions of the same underlying condition. □ □

Remark 5 (Two angles on one truth)**.** *The fixed-point reformulation looks at the dynamics of the rewriting operator T — how the series transforms when iterated. The maximum reformulation looks at the algebraic structure of $s(1-s)$ — the natural product associated with the symmetry $s \to 1-s$.*

They are two observers at different positions in X looking at the same object — exactly as in the Geometry of Complexities [3]. The truth they see is the same: $\mathrm{Im} = 0 \iff \sigma = \frac{1}{2}$.

Remark 6 (The remaining gap)**.** *The Unification Theorem establishes that both reformulations point to the same condition. What remains is to show that $\zeta(s) = 0$* forces *this condition — that zeros cannot exist where* $\mathrm{Im} \neq 0$.

This is the final gap between the reformulation and a complete proof of the Riemann Hypothesis. It is stated precisely as Open Problem 1 below.

7 The Closed Loop Condition

Proposition 4 (Phase factorisation)**.** *In the transformed series, the weight of each term factorises as:*

$$w_n = e^{2\pi t} \cdot n^{-\sigma}$$

The factor $e^{2\pi t}$ is constant — independent of n. It factors out of the sum:

$$\zeta(s) = e^{2\pi t} \sum_{n=1}^{\infty} \frac{e^{-i\phi_n}}{n^\sigma}$$

where the phase is:

$$\phi_n = t \ln n + 2\pi\sigma$$

The phase ϕ_n grows linearly in $\ln n$, with $2\pi\sigma$ as its centre of symmetry. ***[Established]***

Proposition 5 (Closed loop condition). *For $\zeta(s) = 0$, the complex vectors $\frac{e^{-i\phi_n}}{n^\sigma}$ must sum to zero — they must form a* closed loop *in the complex plane.*

A closed loop requires the phases ϕ_n to be distributed symmetrically. The centre of symmetry is $2\pi\sigma$.

When $\sigma = \frac{1}{2}$: the centre is π — a half-turn. The vectors distribute symmetrically. The loop closes.

When $\sigma \neq \frac{1}{2}$: the centre is not π. The distribution is asymmetric. The loop cannot close.

$$\zeta(s) = 0 \iff \textit{phases form closed loop} \iff \sigma = \frac{1}{2}$$

[Sketched] *(step: asymmetric phase distribution prevents loop closure — this is the final gap, Open Problem 1)*

Remark 7 (The complete argument chain). *(i) Rewrite n as complex: $n^{s_n} = n$* ***[Established]***

(ii) Transform $\zeta(s)$ — phases appear ***[Established]***

(iii) Phases centre on $2\pi\sigma$ ***[Established]***

(iv) Centre is π iff $\sigma = \frac{1}{2}$ ***[Established]***

(v) Closed loop requires centre $= \pi$ ***[Sketched]***

(vi) Therefore zeros require $\sigma = \frac{1}{2}$ ***[Open]***

Step (v) is the final gap. Everything else follows.

As a preliminary empirical test, we applied the SymSearch tool [6] to the first ten non-trivial zeros of $\zeta(s)$, measuring their symbolic complexity $S_{\min}$ under a bounded symbolic grammar at depth $d = 3$.

Zero t	$S_{\min}$	**Match**
14.134725	7.16	Yes
21.022040	4.44	Yes
25.010858	1000.00	No
30.424876	1000.00	No
32.935062	1000.00	No
37.586178	7.96	Yes
40.918719	6.58	Yes
43.327073	1000.00	No
48.005151	1000.00	No
49.773832	5.98	Yes

Five zeros are symbolically compressible at $d = 3$ with $S_{\min}$ between 4.44 and 7.96. Five are opaque at this depth.

This preliminary result does not distinguish Riemann zeros from random numbers at this depth — a formal statistical test would require a control group following the protocol of [5]. The result is reported here as an observation, not a conclusion.

8 What This Paper Does Not Claim

This paper does not:

- Prove the Riemann Hypothesis.
- Provide a rigorous proof of Conjecture 3.
- Claim that the empirical SymSearch results establish any structural property of Riemann zeros.

This paper proposes a new reformulation — the critical line as the fixed-point locus of an iterative complex rewriting operator — and presents it as a direction for future rigorous investigation.

9 Open Problems

1. **The final gap: zeros force** $\mathrm{Im} = 0$**. [Open]**
   The Unification Theorem shows that both reformulations reduce to $\mathrm{Im} = 0 \iff \sigma = \frac{1}{2}$. The remaining step is to prove that $\zeta(s) = 0$ implies this condition — that a zero of ζ cannot exist where the imaginary parts do not cancel. This is the precise statement of what remains to be proven.

   Under what conditions does $\prod_{j=1}^{\infty} s_n^{(j)}$ converge? Can convergence to unity be rigorously shown to require $\sigma = \frac{1}{2}$?

2. **Equivalence to the functional equation. [Open]**
   Is the fixed-point condition of Conjecture 3 equivalent to the known functional equation $\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2}) \Gamma(1-s) \zeta(1-s)$? If yes, this reformulation may offer a new path to the hypothesis.

3. **Extension to** L**-functions. [Open]**
   The generalised Riemann Hypothesis concerns all Dirichlet L-functions. Does the iterative rewriting operator generalise naturally to $L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$?

4. **Symbolic complexity of Riemann zeros. [Open]**
   Is the distribution of $S_{\min}$ over the non-trivial zeros of ζ statistically different from the distribution over random numbers? A formal Mann-Whitney test following [5] would answer this.

5. **Fixed-point locus in** X**. [Open]**
   In the Geometry of Complexities [3], the critical line $\mathrm{Re}(s) = \frac{1}{2}$ is the line of maximal complexity symmetry. Can the fixed-point condition of this paper be identified with the isocomplexity condition of the Geometry of Complexities?

10 Summary

Result	Status
Complex rewriting $n^{s_n} = n$	**[Established]**
First transformation of $\zeta(s)$	**[Established]**
Symmetry $s \to 1 - s$ recovered	**[Sketched]**
Fixed-point condition stated	**[Sketched]**
Critical line as maximum of $\mathrm{Re}[s(1-s)]$	**[Established]**
Phase factorisation — $w_n = e^{2\pi t} \cdot n^{-\sigma}$	**[Established]**
Closed loop condition stated	**[Sketched]**
Zeros force $\mathrm{Im} = 0$	**[Open]**
Proof of Riemann Hypothesis	**[Open]**

The central reformulation:

Rewrite each n as a complex number that preserves n.
Iterate the rewriting.
Ask: for which s does the function return to itself?
The answer is the line $\mathrm{Re}(s) = \frac{1}{2}$.
This is where the imaginary parts cancel.
This is where the zeros live.
Proving this rigorously
is the Riemann Hypothesis.

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